A note on post-Riemannian structures of spacetime

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Abstract

A four-dimensional differentiable manifold is given with an arbitrary linear connection $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta \, dx^i$. Megged [1] has claimed that he can define a metric $G_{\alpha\beta}$ by means of a certain integral equation such that the connection is compatible with the metric. We point out that Megged's implicit definition of his metric $G_{\alpha\beta}$ is equivalent to the assumption of a vanishing nonmetricity. Thus his result turns out to be trivial.

In the metric-affine theory of gravitation [2, 3], spacetime is assumed to be a four-dimensional differentiable manifold equipped with a linear connection $\Gamma_\alpha{}^\beta$ and, *independently*, with a metric $g_{\alpha\beta}$. Ne'eman and one of us proposed methods [4, 5] how one could measure the torsion $T^\alpha := D\vartheta^\alpha$ (here ϑ^α is the coframe) and the nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$ of spacetime. In a recent note, Megged [1] has claimed that the use of the nonmetricity $Q_{\alpha\beta}$ is misleading in some sense, since one can define a metric, provided the connection is given, such that it is automatically "connection compatible", if we use his words. We will point out that this rests on the hidden assumption of a vanishing nonmetricity. Since Megged discards the original independent metric $g_{\alpha\beta}$ as a meaningful physical field, there is no way that nonmetricity could enter his theory later.

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1 Metric-affine spacetime

If a metric $g_{\alpha\beta}$ and a connection $\Gamma_\alpha{}^\beta$ are given, for the conventions see [2], then we can raise and lower indices by means of $g_{\alpha\beta}$, such as in $\Gamma_{\alpha\beta} := \Gamma_\alpha{}^\gamma g_{\gamma\beta}$, for example. With the definition of the nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, it is straightforward to compute the symmetric part of the connection (see [6] or [2, eq. (3.10.6)]):

$$2\Gamma_{(\alpha\beta)} = dg_{\alpha\beta} + Q_{\alpha\beta} \ . \tag{1}$$

These are 40 equations, and *no relation* between metric and connection has been assumed.

2 Riemann-Cartan spacetime

If we assume the nonmetricity to vanish, then we find a Riemann-Cartan geometry:

$$Q_{\alpha\beta} = 0 \quad \Rightarrow \quad 2\Gamma_{(\alpha\beta)} = dg_{\alpha\beta} \ . \tag{2}$$

We can choose the coframe to be orthonormal (orthonormal gauge),

$$g_{\alpha\beta} \overset{*}{=} o_{\alpha\beta} := \mathrm{diag}(+1, -1, -1, -1) \ , \tag{3}$$

then

$$\Gamma_{(\alpha\beta)} \overset{*}{=} 0 \quad \text{or} \quad \Gamma_{\alpha\beta} \overset{*}{=} -\Gamma_{\beta\alpha} \ , \tag{4}$$

i. e., we are left with 24 independent components of a Lorentz connection. Thus the connection one-form is $\mathrm{SO}(1,3)$-valued, describing the fact, that the scalar product of two vectors is *invariant* under parallel transport in such a spacetime.

3 Megged's ansatz

He allows only for a connection to be the primary geometrical quantity. Let us call his connection $\hat\Gamma_\alpha{}^\beta$. Then he defines *implicitly* a metric $G_{\alpha\beta} = G_{\beta\alpha}$ by the relation

$$\hat\Gamma_\alpha{}^\gamma G_{\gamma\beta} + \hat\Gamma_\beta{}^\gamma G_{\alpha\gamma} = dG_{\alpha\beta} \ , \tag{5}$$

see his equations [1, eq. (7)] and [1, eq. (8)]. The $G_{\alpha\beta}$, defined by (5), can be taken for raising and lowering indices, such as in $\hat{\Gamma}_{\alpha\beta} := \hat{\Gamma}_\alpha{}^\gamma G_{\gamma\beta}$, for example. Then (5) can be rewritten as

$$2\hat{\Gamma}_{(\alpha\beta)} = dG_{\alpha\beta} \ . \tag{6}$$

If we compare (6) with (1), we recognize that the ansatz (5), which represents 40 independent equations, is equivalent to the assumption

$$\hat{Q}_{\alpha\beta} := -\hat{D}G_{\alpha\beta} = 0 \ . \tag{7}$$

Of course, this can also be seen directly from (5), since

$$dG_{\alpha\beta} - \hat{\Gamma}_\alpha{}^\gamma G_{\gamma\beta} - \hat{\Gamma}_\beta{}^\gamma G_{\alpha\gamma} =: \hat{D}G_{\alpha\beta} \ . \tag{8}$$

In other words, the equation (5), postulated by Megged, amounts to the assumption of a vanishing nonmetricity $\hat{Q}_{\alpha\beta} = 0$. And then it is not surprising to fall back to the metric-compatible Riemann-Cartan spacetime.

References

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