Volume elements of spacetime and a quartet of scalar fields

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Starting with a 'bare' 4-dimensional differential manifold as a model of spacetime, we discuss the options one has for defining a volume element which can be used for physical theories. We show that one has to prescribe a scalar density $\boldsymbol{\sigma}$. Whereas conventionally $\sqrt{|\det g_{ij}|}$ is used for that purpose, with g_{ij} as the components of the metric, we point out other possibilities, namely $\boldsymbol{\sigma}$ as a 'dilaton' field or as a derived quantity from either a linear connection or a quartet of scalar fields, as suggested by Guendelman and Kaganovich. *file volume8.tex, rev., 1998-06-15*

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I. INTRODUCTION

A fundamental premise is that gravity is intimately intertwined with the geometry of spacetime. This geometry is locally characterized by two independent concepts: The concept of a linear connection (parallel transport) and the concept of a metric (length and angle measurements).

Within a gauge approach to gravity the existence of the *linear connection* can be quite satisfactorily explained by the principle of local gauge invariance [1,2]. In contrast to this, it is not clear how to derive the *metric* from some fundamental principle. Usually the existence of the metric is simply assumed, sometimes in disguise of a local gauge group which contains an orthogonal subgroup. Therefore it is natural to ask whether the metric itself is a fundamental quantity, a derived quantity, or a quantity which can be substituted by some more fundamental field. To investigate this question is the main motivation for this article.

In physics it is of fundamental importance to integrate objects on spacetime. This requires the definition of a volume element. We will point out in Sec. II that this definition can be done without reference to any metric. Basically, a volume element can be defined on any differentiable manifold as the determinant of a parallelepiped defined in terms of n vectors, if n is the dimension of the manifold. Then no *absolute* volume measure exists. However, *proportions* of different volumes can be determined. Such a volume element is an (odd) density of weight -1. In order to define an integral, we then need an additional scalar density of weight $+1$.

Usual physical fields are no densities. Therefore the common practice is to take the components of the metric and to build a density according to $\sqrt{|\det g_{ij}|}$. But there exist alternatives which open the gate to alternative theories of gravitation. In this light we will analyze, in Sec. III, one possibility, namely the quartet of scalar fields, as proposed in [3].

II. INTEGRATION ON SPACETIME AND THE VOLUME ELEMENT $\boldsymbol{\sigma}$ AS SCALAR DENSITY

We model spacetime as a 4-dimensional differentiable manifold, which is assumed to be paracompact, Hausdorff, and connected. In order to be able to formulate physical laws on such a spacetime, we have to come up with suitably defined integrals. If we want, for example, to specify a *scalar* action functional W of a physical system,

$$W = \int L = \int \widetilde{\epsilon}\, \widehat{L}\,, \tag{2.1}$$

then, taking the integral in its conventional (Lebesgue) meaning, the Lagrangian L has to be an odd 4-form in order to make the integral (2.1) a scalar. Incidentally, a p-form $\omega = \frac{1}{p!}\,\omega_{i_1\dots i_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}$ is called *even* if it is invariant under a diffeomorphism $x^i \to x'^i(x^j)$ with $\det\left(\partial x^j/\partial x'^i\right) < 0$. It is called *odd* if it changes sign under such a diffeomorphism [4,5,1].

Now, any odd 4-form can be split into a product of a 0-form (or scalar) $\widehat{L}$ and another odd 4-form $\widetilde{\epsilon}$. Let us define a trivial physical system by $\widehat{L} = 1$. Then the integral measures the volume of the corresponding piece of spacetime,

$$\mathrm{Vol} = \int \widetilde{\epsilon}\,. \tag{2.2}$$

For that reason $\widetilde{\epsilon}$ is called a volume form or, more colloquially, a *volume element* of spacetime. This quantity can be split again into two pieces.

As the first piece we have the Levi-Civita $\boldsymbol{\epsilon}$ in mind. Its components $\boldsymbol{\epsilon}_{ijkl}$ are, by definition, *numerically invariant* under diffeomorphisms. We have

$$\boldsymbol{\epsilon} := \frac{1}{4!}\,\boldsymbol{\epsilon}_{ijkl}\, dx^i\wedge dx^j\wedge dx^k\wedge dx^l\,, \qquad \text{with} \quad \boldsymbol{\epsilon}_{0123} = 1 = \text{invariant}\,. \tag{2.3}$$

Consequently $\boldsymbol{\epsilon}$ transforms as an *odd 4-form density of weight* -1 (see, e.g., [1, Appendix A], for details),

$$\boldsymbol{\epsilon}' = \frac{1}{J}\,\boldsymbol{\epsilon} = \frac{\mathrm{sgn}J}{|J|}\,\boldsymbol{\epsilon}\,, \tag{2.4}$$

where $J = \det\big(\partial x^j/\partial x'^i\big)$ is the determinant of the Jacobian matrix of the diffeomorphism $x^i \to x'^i(x^j)$. We are denoting densities by boldface letters.

If we take the interior product $\rfloor$ of an arbitrary frame e_α with the Levi-Civita $\boldsymbol{\epsilon}$ 4-form density, then we find a 3-form $\boldsymbol{\epsilon}_\alpha$; if we contract again, we find a 2-form $\boldsymbol{\epsilon}_{\alpha\beta}$, etc.:

$$\boldsymbol{\epsilon}_\alpha := e_\alpha\rfloor\boldsymbol{\epsilon} = \frac{1}{3!}\,\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^\beta\wedge\vartheta^\gamma\wedge\vartheta^\delta\,, \tag{2.5a}$$

$$\boldsymbol{\epsilon}_{\alpha\beta} := e_\beta\rfloor\boldsymbol{\epsilon}_\alpha = \frac{1}{2!}\,\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^\gamma\wedge\vartheta^\delta\,, \tag{2.5b}$$

$$\boldsymbol{\epsilon}_{\alpha\beta\gamma} := e_\gamma\rfloor\boldsymbol{\epsilon}_{\alpha\beta} = \frac{1}{1!}\,\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^\delta\,, \tag{2.5c}$$

$$\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta} \;= e_\delta\rfloor\boldsymbol{\epsilon}_{\alpha\beta\gamma} = e_\delta\rfloor e_\gamma\rfloor e_\beta\rfloor e_\alpha\rfloor\boldsymbol{\epsilon}\,. \tag{2.5d}$$

Here, the coframe ϑ^β is dual to the frame e_α, that is, $e_\alpha\rfloor\vartheta^\beta = \delta^\beta_\alpha$. The $(\boldsymbol{\epsilon}\,,\boldsymbol{\epsilon}_\alpha\,,\boldsymbol{\epsilon}_{\alpha\beta}\,,\boldsymbol{\epsilon}_{\alpha\beta\gamma}\,,\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta})$ represent a basis for the odd form densities of weight -1. It is called $\boldsymbol{\epsilon}$-basis and can be used to define a metric independent duality operation. Instead of lowering the rank of the $\boldsymbol{\epsilon}$'s, we can also increase their rank by exterior multiplication with the coframe ϑ^μ:

$$\vartheta^\mu\wedge\boldsymbol{\epsilon}_\alpha = +\delta^\mu_\alpha\,\boldsymbol{\epsilon}\,, \tag{2.6a}$$

$$\vartheta^\mu\wedge\boldsymbol{\epsilon}_{\alpha\beta} = -\delta^\mu_\alpha\,\boldsymbol{\epsilon}_\beta + \delta^\mu_\beta\,\boldsymbol{\epsilon}_\alpha\,, \tag{2.6b}$$

$$\vartheta^\mu\wedge\boldsymbol{\epsilon}_{\alpha\beta\gamma} = +\delta^\mu_\alpha\,\boldsymbol{\epsilon}_{\beta\gamma} - \delta^\mu_\beta\,\boldsymbol{\epsilon}_{\alpha\gamma} + \delta^\mu_\gamma\,\boldsymbol{\epsilon}_{\alpha\beta}\,, \tag{2.6c}$$

$$\vartheta^\mu\wedge\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta} = -\delta^\mu_\alpha\,\boldsymbol{\epsilon}_{\beta\gamma\delta} + \delta^\mu_\beta\,\boldsymbol{\epsilon}_{\alpha\gamma\delta} - \delta^\mu_\gamma\,\boldsymbol{\epsilon}_{\alpha\beta\delta} + \delta^\mu_\delta\,\boldsymbol{\epsilon}_{\alpha\beta\gamma}\,. \tag{2.6d}$$

For the $\widetilde{\epsilon}$ (which is an odd 4-form density of weight 0), formulae analogous to (2.5),(2.6) are valid. We have just to add twiddles to the ϵ's.

Since $\boldsymbol{\epsilon}$ is an odd density of weight -1, we can split the volume element $\widetilde{\epsilon}$, if we postulate the existence of an *even scalar density* $\boldsymbol{\sigma}$ of weight $+1$, that is,

$$\boldsymbol{\sigma}' = |J|\,\boldsymbol{\sigma}\,. \tag{2.7}$$

For our purpose here[1], we postulated an *even* scalar density, since the $\widetilde{\epsilon}$ in (2.1) and the Levi-Civita $\boldsymbol{\epsilon}$ in (2.3) are both odd. Then eventually, equation (2.1) can be rewritten as

$$\begin{aligned} W &= \int \underbrace{L}_{\text{odd 4-f.}} = \int \underbrace{\widetilde{\epsilon}}_{\text{odd 4-f.}}\;\underbrace{\widehat{L}}_{\text{scalar}} \\ &= \int \underbrace{\boldsymbol{\epsilon}}_{\substack{\text{odd 4-f. density,}\\ \text{weight } -1}}\;\underbrace{\boldsymbol{\sigma}}_{\substack{\text{even scalar density,}\\ \text{weight } +1}}\;\underbrace{\widehat{L}}_{\text{scalar}}\,. \end{aligned} \tag{2.8}$$

Conventionally, the square root of the modulus of the metric determinant is chosen as the scalar density $\boldsymbol{\sigma}$:

$$_{\mathbf{0}}\boldsymbol{\sigma} := \sqrt{|\det g_{ij}|}\,. \tag{2.9}$$

As soon as a metric $g = g_{ij}\,dx^i\otimes dx^j$ is given—the gravitational potential of general relativity—we can define $_{\mathbf{0}}\boldsymbol{\sigma}$.

Alternatively, we can promote the scalar density to a new fundamental field of nature, compare also the model developed in [1, Sec.6]. The value of such a density $_{\mathbf{1}}\boldsymbol{\sigma}$ can be viewed as a scale factor of the volume element, see also [6,7]. Thus, from a physical point of view, it is interesting to investigate the role of $_{\mathbf{1}}\boldsymbol{\sigma}$ as a scaling parameter which realizes a scale transformation on a physical system.

The possibility to take a metric independent scalar density $_{\mathbf{1}}\boldsymbol{\sigma}$ in place of $\sqrt{|\det g_{ij}|}$ in order to build a proper volume element gives new opportunities to define a concept of scale invariance. In this context, the field $_{\mathbf{1}}\boldsymbol{\sigma}$ is known as a *dilaton field*, which becomes non-trivial in the quantum theory after the conformal (scale) invariance is broken.

In a pure connection ansatz, we prescribe a *linear connection* $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$ (but *no* metric!). Define, as usual, the curvature-2-form by

$$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma\wedge\Gamma_\gamma{}^\beta \tag{2.10}$$

and the Ricci-1-form by

$$\mathrm{Ric}_\alpha := e_\beta\rfloor R_\alpha{}^\beta = \mathrm{Ric}_{i\alpha}\,dx^i\,. \tag{2.11}$$

Then

$$_{\mathbf{2}}\boldsymbol{\sigma} := \sqrt{|\det \mathrm{Ric}_{ij}|}\,, \tag{2.12}$$

with $\mathrm{Ric}_{ij} = \mathrm{Ric}_{i\alpha}e_j{}^\alpha$, is a viable scalar density, as first suggested by Eddington [8, Sec. 92], compare also Schrödinger [9] and the more recent work of Kijowski and collaborators, see [10]. Likewise, the corresponding quantity based on the *symmetric* part of the Ricci tensor,

$$_{\mathbf{2'}}\boldsymbol{\sigma} := \sqrt{|\det \mathrm{Ric}_{(ij)}|}\,, \tag{2.13}$$

also qualifies as a volume measure. Note that $_{\mathbf{2}}\boldsymbol{\sigma}$ and $_{\mathbf{2'}}\boldsymbol{\sigma}$ may have singular points in such a theory as soon as the Ricci tensor or its symmetric part vanish. There seems to exist no criterion around which would prefer, say, $_{\mathbf{2}}\boldsymbol{\sigma}$, as compared to $_{\mathbf{2'}}\boldsymbol{\sigma}$. Lately, such theories have been abandoned.

In contrast, the non-linear electrodynamics of *Born-Infeld* [11] with its structurally similar Lagrangian (f = maximal field strength)

$$_{\mathbf{3}}\boldsymbol{\sigma} = \sqrt{|\det(g_{ij}+F_{ij}/f)|} - \sqrt{|\det g_{ij}|}\,, \tag{2.14}$$

has been in the focus of renewed interest, see [12].

[1] In reference [1, Eq.(A.1.33)] we took an *odd* scalar density instead, which we also denoted by the same letter σ.

III. THE QUARTET THEORY

A further possibility is the introduction of a *quartet of scalar fields*. First, in close analogy to the components $\boldsymbol{\epsilon}_{ijkl}$ of the Levi-Civita $\boldsymbol{\epsilon}$, we can define the totally antisymmetric tensor density $\boldsymbol{\epsilon}^{ijkl}$ of weight +1. We put its numerically invariant component $\boldsymbol{\epsilon}^{0123} = -1$. Then we define [3]

$$\begin{aligned} {}_4\boldsymbol{\sigma} &:= -\boldsymbol{\epsilon}^{ijkl} \left(\partial_i\varphi^{(0)}\right)\left(\partial_j\varphi^{(1)}\right)\left(\partial_k\varphi^{(2)}\right)\left(\partial_l\varphi^{(3)}\right) \\ &= -\frac{1}{4!}\,\boldsymbol{\epsilon}^{ijkl}\,\epsilon_{ABCD}\,\left(\partial_i\varphi^A\right)\left(\partial_j\varphi^B\right)\left(\partial_k\varphi^C\right)\left(\partial_l\varphi^D\right)\,, \end{aligned} \tag{3.1}$$

where $A,\ldots,D$ are indices of interior space. This definition yields, for the volume 4-form

$$\overline{\eta} := {}_4\boldsymbol{\sigma}\,\boldsymbol{\epsilon}\,, \tag{3.2}$$

the following relations:

$$\begin{aligned} \overline{\eta} &= d\varphi^{(0)}\wedge d\varphi^{(1)}\wedge d\varphi^{(2)}\wedge d\varphi^{(3)} \\ &= \frac{1}{4!}\,\epsilon_{ABCD}\,d\varphi^A\wedge d\varphi^B\wedge d\varphi^C\wedge d\varphi^D\,. \end{aligned} \tag{3.3}$$

If we introduce the abbreviation

$$\partial_A := \frac{\partial}{\partial\varphi^A}\,, \tag{3.4}$$

then the duality of $d\varphi^A$ and ∂_B can be expressed as follows:

$$d\varphi^A\,[\partial_B] = \delta^A_B\,. \tag{3.5}$$

In analogy to the set of equation (2.5), we define the 3-form and the 2-form

$$\overline{\eta}_A := \partial_A\rfloor\overline{\eta}\,,\quad \overline{\eta}_{AB} := \partial_B\rfloor\overline{\eta}_A\,,\quad \text{etc.} \tag{3.6}$$

Explicitly they read

$$\overline{\eta}_A = \frac{1}{3!}\epsilon_{ABCD}d\varphi^B\wedge d\varphi^C\wedge d\varphi^D\,, \tag{3.7a}$$

$$\overline{\eta}_{AB} = \frac{1}{2!}\epsilon_{ABCD}d\varphi^C\wedge d\varphi^D\,,\quad \text{etc.} \tag{3.7b}$$

In analogy to (2.6) we have

$$d\varphi^N\wedge\overline{\eta}_A = +\delta^N_A\,\overline{\eta}\,,\quad d\varphi^N\wedge\overline{\eta}_{AB} = -\delta^N_A\,\overline{\eta}_B + \delta^N_B\,\overline{\eta}_A\,, \tag{3.8}$$

and so on. We contract (3.8) and find

$$\overline{\eta} = \frac{1}{4}\,d\varphi^N\wedge\overline{\eta}_N\,,\quad \overline{\eta}_A = \frac{1}{3}\,d\varphi^N\wedge\overline{\eta}_{AN}\,,\quad \text{etc.} \tag{3.9}$$

We differentiate (3.9):

$$d\,\overline{\eta} = -\frac{1}{4}\,d\varphi^N\wedge d\,\overline{\eta}_N\,,\quad d\,\overline{\eta}_A = -\frac{1}{3}\,d\varphi^N\wedge d\,\overline{\eta}_{AN}\,,\ \text{etc.} \tag{3.10}$$

Now, $\overline{\eta}$, as a 4-form, is closed:

$$d\,\overline{\eta} = 0\,. \tag{3.11}$$

Provided $d\varphi^A \neq 0$, we find successively,

$$d\,\overline{\eta}_A = 0\,,\quad d\,\overline{\eta}_{AB} = 0\,,\quad \text{etc.} \tag{3.12}$$

Using this information, we can partially integrate (3.9) and can prove that all these forms are not only closed, but also exact:

$$\overline{\eta} = d\left[\frac{1}{4}\,\varphi^N\wedge\overline{\eta}_N\right]\,,\quad \overline{\eta}_A = d\left[\frac{1}{3}\,\varphi^N\wedge\overline{\eta}_{AN}\right]\,,\quad \text{etc.} \tag{3.13}$$

Using (3.3) and (3.7), we find

$$\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A} = \overline{\eta}_A\,,\quad \frac{\partial\,\overline{\eta}_A}{\partial\,d\varphi^B} = \overline{\eta}_{AB}\,,\quad \text{etc.} \tag{3.14}$$

or, because of (3.13):

$$d\,\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A} = 0\,,\quad d\,\frac{\partial\,\overline{\eta}_A}{\partial\,d\varphi^B} = 0\,,\quad \text{etc.} \tag{3.15}$$

Since the corresponding "forces" vanish too, as can be seen from (3.3) and (3.7),

$$\frac{\partial\,\overline{\eta}}{\partial\,\varphi^A} = 0\,,\quad \frac{\partial\,\overline{\eta}_A}{\partial\,\varphi^B} = 0\,,\quad \text{etc.,} \tag{3.16}$$

we find an analogous result for the variational derivatives:

$$\frac{\delta\,\overline{\eta}}{\delta\,\varphi^A} = 0\,,\quad \frac{\delta\,\overline{\eta}_A}{\delta\,\varphi^B} = 0\,,\quad \text{etc.} \tag{3.17}$$

Similarly, we have

$$\frac{\delta\overline{\eta}}{\delta\vartheta^\alpha} = 0\,,\quad \frac{\delta\overline{\eta}_A}{\delta\vartheta^\alpha} = 0\,,\quad \text{etc.} \tag{3.18}$$

and

$$\frac{\delta\overline{\eta}}{\delta g_{\alpha\beta}} = 0\,,\quad \frac{\delta\overline{\eta}_A}{\delta g_{\alpha\beta}} = 0\,,\quad \text{etc.} \tag{3.19}$$

That the volume element is an exact form is the distinguishing feature of this quartet ansatz. Under these circumstances, the volume (2.2) can be expressed, via Stokes' theorem, as a 3-dimensional surface integral which doesn't contribute to the variation of the action functional.

Using Eq. (2.8) and the volume element (3.3) and denoting the gravitational Lagrangian by $V = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta)$ and the matter Lagrangian by $L_{\rm m} = L_{\rm m}(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta, \Psi, d\Psi)$, the action W reads

$$W = \int (V + L_{\rm m}) = \int \overline{\eta}\ \underbrace{(\widehat{V} + \widehat{L}_{\rm m})}_{\rm scalar} = \int d\,\chi\,(\widehat{V} + \widehat{L}_{\rm m})\ , \tag{3.20}$$

see Guendelman and Kaganovich [3]. Note that the 3-form χ, according to (3.13), explicitly reads $\chi := \varphi^N \wedge \overline{\eta}_N/4$. If we add a constant λ to the scalar Lagrangian, we find

$$\int d\,\chi\,(\widehat{V} + \widehat{L}_{\rm m} + \lambda) = W + \lambda \int d\,\chi\ . \tag{3.21}$$

Since the 3-dimensional hypersurface integral $\int_{\partial\,\rm Vol}\chi$ doesn't contribute to the variation, the scalar Lagrangian is invariant under the addition of a constant.

Variation with respect to φ^A yields the corresponding field equations

$$\frac{\partial\,(V + L_{\rm m})}{\partial\varphi^A} - d\,\frac{\partial\,(V + L_{\rm m})}{\partial\,d\varphi^A} = 0\ . \tag{3.22}$$

Suppose, see [3], that $\widehat{V}$ and $\widehat{L}_{\rm m}$ do *not* depend on the quartet field at all,

$$\frac{\partial\,\widehat{V}}{\partial\,\varphi^A} = 0\ ,\quad \frac{\partial\,\widehat{V}}{\partial\,d\varphi^A} = 0\ ,\qquad \frac{\partial\,\widehat{L}_{\rm m}}{\partial\,\varphi^A} = 0\ ,\quad \frac{\partial\,\widehat{L}_{\rm m}}{\partial\,d\varphi^A} = 0\ , \tag{3.23}$$

then the field equations for the quartet field read

$$(\widehat{V} + \widehat{L}_{\rm m})\,\frac{\partial\,\overline{\eta}}{\partial\varphi^A} - d\Big[(\widehat{V} + \widehat{L}_{\rm m})\,\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}\Big] = 0\ . \tag{3.24}$$

The first term vanishes, since $\overline{\eta}$ does not depend on φ^A explicitly, see (3.16). Then the Leibniz rule yields

$$d\,\Big[(\,\widehat{V} + \widehat{L}_{\rm m})\,\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}\Big] \tag{3.25}$$

$$= \frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}\,d\,(\widehat{V} + \widehat{L}_{\rm m}) + (\widehat{V} + \widehat{L}_{\rm m})\ \underbrace{d\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}}_{\stackrel{(3.15)}{=}0} = 0\ . \tag{3.26}$$

Provided $\varphi^A \neq 0$ and $d\varphi^A \neq 0$, we can conclude that

$$d\,(\widehat{V} + \widehat{L}_{\rm m}) = 0\ ,\qquad \text{i.e.,}\qquad \widehat{V} + \widehat{L}_{\rm m} = \text{const}\ . \tag{3.27}$$

The gravitational field equations following from $\delta g_{\alpha\beta}$ and $\delta\Gamma_\alpha{}^\beta$ are *not* disturbed by the existence of φ^A. Hence the usual metric-affine formalism applies in its conventional form (see [1], for recent developments cf. [13–16]), but the field equation (3.27) for the scalar field quartet φ^A has to be appended. Perhaps surprisingly, it is only one equation since, in addition to (3.18) and (3.19), we trivially have

$$\frac{\delta\overline{\eta}}{\delta\Gamma_\alpha{}^\beta} = 0\ ,\quad \frac{\delta\overline{\eta}_A}{\delta\Gamma_\alpha{}^\beta} = 0\ ,\quad \text{etc.} \tag{3.28}$$

IV. CONCLUSION

We transparently displayed the necessary structures for building up a volume element and pointed out several physical alternatives to the usual metric volume element, giving rise to different gravity theories. Within the framework of metric–affine gravity we can reproduce the essential features of the Guendelman-Kaganovich theory without the necessity to specify the gravitational first-order Lagrangian other than by the property (3.23).

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